Utility maximization with cobb-douglas

Consider a consumer who derives utility from consuming two goods, x_1 and x_2. The consumer's utility function is given by:

$$U(x_1, x_2) = x_1^{\alpha} x_2^{\beta}, \tag{1}$$

where α and β are positive constants. The consumer has a budget constraint given by:

$$p_1 x_1 + p_2 x_2 = M, \tag{2}$$

where p_1 and p_2 are the prices of good 1 and good 2, respectively, and M is the consumer's income. Find the optimal consumption bundle and the optimal utility for the consumer.

Solution

Lagrangian function

To solve this problem, we will set up the Lagrangian function, which includes the utility function and the budget constraint:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{\beta} + \lambda(M - p_1 x_1 - p_2 x_2). \tag{3}$$

First-order conditions

Next, we find the first-order conditions for maximizing the Lagrangian function with respect to x_1, x_2, and λ:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta} - \lambda p_1 = 0, \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1} - \lambda p_2 = 0, \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - p_1 x_1 - p_2 x_2 = 0. \tag{6}$$

Optimal consumption bundle

Now, we will solve the first two first-order conditions to eliminate λ and find the optimal consumption bundle:

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{p_1} = \frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2}. \tag{7}$$

Simplifying the equation and rearranging terms, we get:

$$\frac{x_1}{x_2} = \frac{\alpha p_2}{\beta p_1}. \tag{8}$$

$$x_1 = \frac{\alpha p_2}{\beta p_1} x_2. \tag{9}$$

Now, we substitute the expression for the optimal consumption ratio back into the budget constraint:

$$p_1 \left(\frac{\alpha p_2}{\beta p_1}\right) x_2 + p_2 x_2 = M \tag{10}$$

Solving for x_2, we find:

$$x_2^* = \frac{M}{p_2} \frac{\beta}{\beta + \alpha} \tag{11}$$

Now, we can find the optimal x_1^* using the expression for the optimal consumption ratio:

$$x_1^* = \frac{\alpha p_2}{\beta p_1} x_2^* = \frac{M}{p_1} \frac{\alpha}{\beta + \alpha} \tag{12}$$

Optimal utility

Finally, we substitute the optimal consumption bundle into the utility function to find the optimal utility:

$$U^* = U(x_1^*, x_2^*) = \left(\frac{M}{p_1} \frac{\alpha}{\beta + \alpha}\right)^{\alpha} \left(\frac{M}{p_2} \frac{\beta}{\beta + \alpha}\right)^{\beta}. \tag{13}$$